Portfolio selection by minimizing the present value of capital injection costs

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Abstract

This paper considers the portfolio selection and capital injection problem for a diffusion risk model within the classical Black-Scholes financial market. It is assumed that the original surplus process of an insurance portfolio is described by a drifted Brownian motion, and that the surplus can be invested in a risky asset and a risk-free asset. When the surplus hits zero, the company can inject capital to keep the surplus positive. In addition, it is assumed that both fixed and proportional costs are incurred upon each capital injection. Our objective is to minimize the expected value of the discounted capital injection costs by controlling the investment policy and the capital injection policy. We first prove the continuity of the value function and a verification theorem for the corresponding HJB equation. We then show that the optimal investment policy is a solution to a terminal value problem of an ordinary differential equation. In particular, explicit solutions are derived in some special cases and a series solution is obtained for the general case. Also, we propose a numerical method to solve the optimal investment and capital injection policies. Finally, a numerical study is carried out to illustrate the effect of the model parameters on the optimal policies.

Keywords Backward Euler method; Capital injection; HJB equation; Portfolio selection; Transaction costs

1 Introduction

In the actuarial and mathematical insurance literatures, the portfolio selection problem for a diffusion risk model within the classical Black-Scholes financial market has attracted great interest in the past few decades. For example, Browne (1995) studied the optimal investment policy that maximizes the utility of terminal wealth and minimizes the ruin probability; Bai and Guo (2008) derived the optimal reinsurance and investment policies for the same problem with multiple risky assets; Promislow and Young (2005) and Luo et al. (2008) considered the problem of minimizing the ruin probability with reinsurance and investment controls. Furthermore, the portfolio selection problem for a diffusion risk model with dividend payments has been studied by many authors. Among them, Højgaard and Taksar (2001) studied the optimal dividend problem in a diffusion risk model with stochastic investment returns; and Højgaard and Taksar (2004) considered the optimal reinsurance, investment and dividend problem with the objective of maximizing the expected value of total discounted dividend payments prior to the ruin time. For more details about the portfolio selection problem in modern risk theory, we refer the readers to Liu and Yang (2004), Zhang and Siu (2009), Schmidli (2008), and references therein.

In recent years, capital injection is another key factor to consider in stochastic control problems. The idea of capital injection is to keep the company's surplus process above some fixed level. For an insurance company, an obvious advantage of injecting capital is to avoid the event of ruin. As a result, the total amount of capital injection can be treated as the cost of keeping the company alive in the market. From this view point, the expected value of the total discounted capital injection costs, in addition to the ruin probability and the expected value of total discounted dividend payments, can serve as another important objective to optimize. In practice, the issue of capital injection is also of great interest to a mutual insurance company that is not interested in profits but would like to minimize the cost associated with raising additional capital. Recent research in this direction can be found in Sethi and Taksar (2002), Løkka and Zervos (2008), Kulenko and Schimidli (2008), Paulsen (2008), Eisenberg and Schmidli (2009), Yao et al. (2010), Scheer and Schimidli (2011), Zhou and Yuen (2012), and references therein.

Inspired by these literatures, we study the optimization problem of portfolio selection and capital injection for an insurance company by minimizing the expected value of the total discounted capital injection costs. In the model set-up of previous works, we note that there exists only a risky asset but without a risk-free asset in the financial market, for example, see Browne (1995) and Eisenberg

(2010). Specifically, it was assumed that part of a company's surplus can be invested in the risky asset, and that the remaining surplus is kept in cash. With this assumption, it was shown that the optimal amount invested in the risky asset is constant and does not change with the surplus value or the time. In the present paper, a few more realistic features are considered. The first feature is to include not only proportional costs but also a fixed cost for capital injection. The second is to incorporate a risk-free asset into the financial market as it is a very important investment tool, especially for an insurance company. In this model set-up, we obtain a semi-explicit optimal investment policy in series form when both risk-free interest rate and discount rate are positive. The optimal investment policy is no longer a constant value. Furthermore, the optimal amount of capital injection can be explicitly obtained.

The rest of the paper is organized as follows. In Section 2, we present the formulation of the stochastic control problem of this study. In Section 3, we prove the continuity of the value function and the verification theorem for the solution to the HJB equation. In Section 4, we derive the solution to the HJB equation. Finally, in Section 5, we carry out a numerical study to assess the impact of the model parameters on the optimal investment and capital injection policies.

2 Model formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space where the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Throughout the paper, it is assumed that all stochastic processes and random variables are well defined on this probability space.

Consider the classical Black-Scholes market, which includes one risk-free asset and one risky asset. The price process of the risk-free asset $\{P_t\}$ is given by

$$dP_t = r_0 P_t dt, \qquad t > 0,$$

where $r_0 \geq 0$ is the risk-free interest rate. The price process of the risky asset $\{S_t\}$ follows a geometric Brownian motion given by

$$\frac{dS_t}{S_t} = \alpha dt + \beta dB_t^S, \qquad t > 0,$$

where α and β are positive constants, and $\{B_t^S\}$ is a standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathbb{P}$. As usual, we assume that there is a positive market-risk premium which implies $r_0 < \alpha$.

Suppose that the surplus process before investments and injections of a large insurance company follows a diffusion process $\{X_t\}$ which has the form

$$dX_t = \mu dt + \sigma dB_t^X,\ t > 0, \qquad \text{with } X_0 = x, \tag{2.1}$$

where $\mu \geq 0$ is the profit rate, $\sigma > 0$ is the volatility, and the process $\{B_t^X\}$ is also a standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathbb{P}$. It is assumed that the two Brownian motions $\{B_t^S\}$ and $\{B_t^X\}$ are dependent with a correlation coefficient ρ. Therefore, we have $|\rho| \leq 1$ and $\mathbb{E}[B_t^S B_t^X] = \rho t$ for $t > 0$.

We now consider a portfolio selection and capital injection problem. Assume that the company's surplus can be invested in the Black-Scholes market when it is positive. Let $\pi = \{\pi_t, t \geq 0\}$ be an investment policy where π_t denotes the amount of money invested in the risky asset at time t. Note that the remaining part of the surplus is put in the risk-free asset. We further assume that there is no transaction cost for the investment, and that there is no restriction on the investment policy (that is, both short selling of the risky asset and borrowing money at the risk-free interest rate are allowed). In addition, to avoid ruin, capital injection can be made in the impulse form with a linear cost function when the company's surplus hits zero.[1] For each capital injection, there exist transaction costs which include a fixed cost $K > 0$ and a proportional cost rate $l \geq 1$. Thus, if the capital c is injected at time t, then the total capital injection cost $K + lc$ occurs at time t.

Consider an investment policy π and a capital injection policy c. The company's surplus of (2.1) at time $t > 0$ can be rewritten as

$$X_t^{\pi,c} = x + \mu t + \sigma B_t^X + \int_0^t \left[r_0 X_{s-}^{\pi,c} + (\alpha - r_0)\pi_s\right] ds + \int_0^t \beta \pi_s dB_s^S + \sum_{0 \leq s \leq t} c I_{\{X_{s-}^{\pi,c}=0\}}, \tag{2.2}$$

with $X_{0-}^{\pi,c} = x$. The pair of policies (π, c) is said to be admissible if

(1) π is a predictable process with respect to $\{\mathcal{F}_t\}$ such that, for any fixed $T > 0$,

$$\int_0^T \mathbb{E}[\pi_s^2] ds < \infty; \tag{2.3}$$

(2) c is a constant such that $0 < c < \infty$;

(3) the stochastic differential equation (2.2) has a unique strong solution.

[1] Here, we fix the surplus level for capital injection at zero. For a model with flexible surplus level for capital injection, we refer the readers to Sethi and Taksar (2002).

Denote by $\mathcal{Z}$ the set of all admissible policies. Note that, for each pair of Markov control policies $(\pi, c) \in \mathcal{Z}$, the surplus process $\{X_t^{\pi,c}\}$ determined by the stochastic differential equation (2.2) is a strong Markov process. Associated with $\{X_t^{\pi,c}\}$, we define an operator $\mathcal{A}^\pi$ such that, for any twice continuously differentiable function $f(x)$,

$$\mathcal{A}^\pi f(x) = \frac{1}{2}[\beta^2\pi^2 + 2\rho\beta\sigma\pi + \sigma^2]f''(x) + [r_0 x + (\alpha - r_0)\pi + \mu]f'(x), \quad x > 0.$$

In this paper, we aim at minimizing the present value of the total capital injection costs in the presence of investment and capital injection policies. Note that the controlled process (2.2) only involves the control variables (π, c) and the state of controlled process $X_t^{\pi,c}$ but not the state variables S_t. So, the performance function associated with this stochastic control problem is defined as

$$V^{\pi,c}(x) = \mathbb{E}\left[\sum_{t\geq 0} e^{-\delta t}(K + lc)I_{\{X_{t-}^{\pi,c}=0\}}|X_{0-}^{\pi,c} = x\right], \quad x > 0,$$

for any $(\pi, c) \in \mathcal{Z}$, where $\delta \geq 0$ is a constant discount rate. As a result, the value function is given by

$$V(x) = \inf_{(\pi,c)\in\mathcal{Z}} V^{\pi,c}(x), \quad x > 0. \tag{2.4}$$

Our objective is to find a pair of optimal policies $(\pi^*, c^*) \in \mathcal{Z}$ such that $V(x) = V^{\pi^*,c^*}(x)$, which leads to a regular-impulse mixed stochastic control problem. Note that the optimization problem remains the same when a multiplier $1/l$ is applied to the performance function. Thus, without loss of generality, we set $l = 1$.

3 Continuity of the value function and verification theorem

In this section, we prove continuity of the value function and derive the boundary conditions for the value function at zero and infinity. In addition, by employing the dynamic programming principle, we provide the HJB equation satisfied by the value function. We also present the verification theorem for the solution to the HJB equation with the corresponding boundary conditions.

Firstly consider the company's initial surplus process $\{X_t\}$ given by (2.1). We define the first hitting time $\tau_y^X = \inf\{t \geq 0; X_t = y\}$. In addition, we denote ρ_+ and ρ_- as the positive and negative root, respectively, to the equation: $\frac{1}{2}\sigma^2 x^2 + \mu x - \delta = 0$. That is,

$$\rho_+ = \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2} \quad \text{and} \quad \rho_- = \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}.$$

Associated with the stochastic process $\{X_t\}$, we have the following result.

Lemma 3.1. *For any $y \geq x \geq 0$, it follows that*

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta\tau_y^X} I_{\{\tau_y^X<\tau_0^X\}}|X_0=x\right] &= \frac{e^{\rho_+ x}-e^{\rho_- x}}{e^{\rho_+ y}-e^{\rho_- y}}, &(3.1)\\
\mathbb{E}\left[e^{-\delta\tau_0^X} I_{\{\tau_0^X<\tau_y^X\}}|X_0=x\right] &= \frac{e^{\rho_+ y}e^{\rho_- x}-e^{\rho_+ x}e^{\rho_- y}}{e^{\rho_+ y}-e^{\rho_- y}}, &(3.2)\\
\mathbb{E}\left[e^{-\delta\tau_0^X\wedge\tau_y^X}|X_0=x\right] &= \frac{1-e^{\rho_- y}}{e^{\rho_+ y}-e^{\rho_- y}}e^{\rho_+ x}+\frac{e^{\rho_+ y}-1}{e^{\rho_+ y}-e^{\rho_- y}}e^{\rho_- x}, &(3.3)
\end{aligned}$$

and, for any $x \geq y \geq 0$, it follows that

$$\mathbb{E}\left[e^{-\delta\tau_y^X}|X_0=x\right] = e^{\rho_-(x-y)}. \tag{3.4}$$

Proof. The results (3.1)–(3.3) are directly given by Borodin and Salminen (2002, page 233). In addition, (3.4) is the limiting case of (3.2) by letting $y \to \infty$, and then replacing x by $x-y$. □

Employing Lemma 3.1, we can give an upper bound for the value function $V(x)$, which does not depend on the risk-free rate r_0.

Proposition 3.1. *The value function $V(x)$ of (2.4) is bounded such that*

$$V(x) \leq \frac{K+c^*}{1-e^{\rho_- c^*}}e^{\rho_- x}, \quad x>0, \tag{3.5}$$

where $c^*>0$ is the unique solution to the equation

$$1-e^{\rho_- c}+(K+c)\rho_- e^{\rho_- c}=0. \tag{3.6}$$

Proof. To give an upper bound for $V(x)$, we consider a special investment policy π such that $\pi_t=0$ for all $t \geq 0$. In this case, we note that the surplus process with $r_0>0$ is always larger than that with $r_0=0$, so the discounted capital injection with $r_0>0$ is always less than that with $r_0=0$. Thus, in the special case that $\pi_t=0$ for all $t\geq 0$ and $r_0=0$, $V^{\pi,c}(x)$ is an upper bound for $V(x)$ of (2.4).

In this special case, according to the strong Markov property, the performance function $V^{\pi,c}(x)$ satisfies

$$\begin{aligned}
V^{\pi,c}(x) &= \mathbb{E}\left[e^{-\delta\tau_c^X}|X_0=x\right]V^{\pi,c}(c), \quad x\geq c,\\
V^{\pi,c}(x) &= \mathbb{E}\left[e^{-\delta\tau_0^X}|X_0=x\right](K+c+V^{\pi,c}(c)), \quad x\geq 0.
\end{aligned}$$

These together with (3.4) yield

$$V^{\pi,c}(x)=\frac{K+c}{1-e^{\rho_- c}}e^{\rho_- x}, \quad x\geq c. \tag{3.7}$$

On the other hand, for $0 \leq x \leq c$, we have

$$\begin{aligned} V^{\pi,c}(x) &= \mathbb{E}\left[e^{-\delta\tau_0^X \wedge \tau_c^X}|X_0 = x\right] V^{\pi,c}(c) + \mathbb{E}\left[e^{-\delta\tau_0^X} I_{\{\tau_0^X < \tau_c^X\}}|X_0 = x\right](K + c) \\ &= \frac{K + c}{1 - e^{\rho_- c}} e^{\rho_- x}, \end{aligned} \tag{3.8}$$

where the last equality follows by direct calculations using (3.2), (3.3) and (3.7). Thus, according to (3.7) and (3.8), it follows that

$$V(x) \leq \inf_{c \geq 0} V^{\pi,c}(x) = \inf_{c \geq 0} \frac{K + c}{1 - e^{\rho_- c}} e^{\rho_- x}, \quad x \geq 0. \tag{3.9}$$

Taking the first derivative with respect to c yields that the minimizer $c = c^*$ should be the solution to (3.6). In addition, the left hand side of (3.6) is an increasing function with respect to c, and is negative for $c = 0$ and positive as $c \to \infty$. Thus, the solution c^* to (3.6) is unique. □

Proposition 3.2. *The value function $V(x)$ of (2.4) is a continuous and strictly decreasing function.*

Proof. We first define a special surplus process $\{Y_t\}$ such that $dY_t = (r_0 Y_t + \mu)dt + \sigma dB_t^X$, and its first hitting time $\tau_y^Y = \inf\{t \geq 0; Y_t = y\}$.

For any $x > 0$, $h > 0$ and given an admissible policy π, we define a new admissible policy $\tilde{\pi}$ such that

$$\tilde{\pi}_t = \begin{cases} 0, & 0 \leq t < \tau_x^Y, \\ \pi_{t-\tau_x^Y}, & t \geq \tau_x^Y, \end{cases}$$

where τ_x^Y is the first hitting time of the surplus process $\{Y_t\}$ with $Y_0 = x + h$. Then, employing the strong Markov property, we obtain

$$V^{\tilde{\pi},c}(x + h) = \mathbb{E}[e^{-\delta\tau_x^Y}|Y_0 = x + h] V^{\pi,c}(x).$$

Taking the infimum in the admissible set gives

$$V(x + h) \leq \inf_{(\pi,c)\in\mathcal{Z}} V^{\tilde{\pi},c}(x + h) = \mathbb{E}[e^{-\delta\tau_x^Y}|Y_0 = x + h] V(x) < V(x),$$

which implies that $V(x)$ is a strictly decreasing function.

Also, for any $x > 0$, $h > 0$ and given an admissible policy π for initial surplus $x + h$, we define a new admissible policy $\hat{\pi}$ such that

$$\hat{\pi}_t = \begin{cases} 0, & 0 \leq t < \tau_{x+h}^Y, \\ \pi_{t-\tau_{x+h}^Y}, & t \geq \tau_{x+h}^Y, \end{cases}$$

where τ_{x+h}^Y is the first hitting time of the surplus process $\{Y_t\}$ with $Y_0 = x$. Again, employing the strong Markov property, we obtain

$$\begin{aligned} V^{\hat{\pi},c}(x) &= \mathbb{E}\left[e^{-\delta\tau_{x+h}^Y} I_{\{\tau_{x+h}^Y < \tau_0^Y\}} | Y_0 = x\right] V^{\pi,c}(x+h) \\ &\quad + \mathbb{E}\left[e^{-\delta\tau_0^Y} I_{\{\tau_0^Y < \tau_{x+h}^Y\}} | Y_0 = x\right] (K + c + V^{\hat{\pi},c}(c)). \end{aligned} \tag{3.10}$$

Note that, $\{X_t\}$ is defined by (2.1) and $\{Y_t\}$ is defined at the beginning of the proof. Given the same initial value $X_0 = x$ and $Y_0 = x$, it follows that $\tau_{x+h}^Y \leq \tau_{x+h}^X$, $\tau_0^Y \geq \tau_0^X$ and hence $\{\tau_0^Y < \tau_{x+h}^Y\} \subseteq \{\tau_0^X < \tau_{x+h}^X\}$. Thus, according to (3.2) and (3.5), it follows that

$$\begin{aligned} 0 \leq &\ \lim_{h\to 0} \mathbb{E}\left[e^{-\delta\tau_0^Y} I_{\{\tau_0^Y < \tau_{x+h}^Y\}} | Y_0 = x\right] (K + c + V^{\hat{\pi},c}(c)) \\ \leq &\ \lim_{h\to 0} \mathbb{E}\left[e^{-\delta\tau_0^X} I_{\{\tau_0^X < \tau_{x+h}^X\}} | X_0 = x\right] (K + c + V^{\hat{\pi},c}(c)) \quad = 0. \end{aligned}$$

Then, taking infimum in the admissible set on both sides of (3.10) yields

$$V(x) \leq \mathbb{E}\left[e^{-\delta\tau_{x+h}^Y} I_{\{\tau_{x+h}^Y < \tau_0^Y\}} | Y_0 = x\right] V(x+h) + \mathcal{O}(h),$$

where $\mathcal{O}(h)$ denotes infinitesimal of the same order with h. Combining this with (3.1) and (3.5) and using the same arguments above, we have

$$\begin{aligned} 0 < V(x) - V(x+h) &\leq \left(1 - \mathbb{E}\left[e^{-\delta\tau_{x+h}^Y} I_{\{\tau_{x+h}^Y < \tau_0^Y\}} | Y_0 = x\right]\right) V(x+h) + \mathcal{O}(h) \\ &\leq \left(1 - \mathbb{E}\left[e^{-\delta\tau_{x+h}^X} I_{\{\tau_{x+h}^X < \tau_0^X\}} | X_0 = x\right]\right) V(x+h) + \mathcal{O}(h) \\ &\leq \left(1 - \frac{e^{(\rho_+ - \rho_-)x} - 1}{e^{(\rho_+ - \rho_-)(x+h)} - 1} e^{-\rho_- h}\right) \frac{K + c^*}{1 - e^{\rho_- c^*}} e^{\rho_-(x+h)} + \mathcal{O}(h) \\ &= \mathcal{O}(h), \end{aligned}$$

which means that $V(x)$ is continuous from the right. Along the same lines, we can show $V(x)$ is also continuous from the left. Hence, we conclude that $V(x)$ is a continuous function. $\square$

In the following proposition, the boundary conditions for the value function are given.

Proposition 3.3. *The value function $V(x)$ of (2.4) satisfies*

$$\lim_{x\to\infty} V(x) = 0, \tag{3.11}$$

and

$$V(0) = \min_{c>0} \{V(c) + K + c\}. \tag{3.12}$$

Proof. For the special investment policy $\pi_t = 0$ for all $t \geq 0$, it follows from Gerber and Yang (2007) that the surplus process will never lead to ruin as $x \to \infty$, so capital injection will never happen in this case. Hence, (3.11) follows. In addition, the boundary condition (3.11) can also be directly derived from (3.5).

Note that capital injection is a compulsory action when the surplus $x = 0$. If c is the amount of capital injection, then the surplus increases from 0 to c, and the value function takes a value of $V(c) + K + c$. According to the definition of the value function, the amount of capital injection c should minimize $V(c) + K + c$ for $c > 0$. Thus, (3.12) follows. □

Since the value function $V(x)$ has been proved to be continuous, the following dynamic programming principle holds:

$$V(x) = \inf_{(\pi,c)\in\mathcal{Z}} \mathbb{E}\left[e^{-\delta(\tau\wedge\tau_0)} V\left(X^{\pi,c}_{(\tau\wedge\tau_0)-}\right) | X^{\pi,c}_{0-} = x\right], \tag{3.13}$$

for any $x > 0$ and a stopping time τ of the filtration $\{\mathcal{F}_t\}$, where τ_0 is the first time that the process $\{X^{\pi,c}_t\}$ hits zero, i.e, $\tau_0 = \inf\{t \geq 0; X^{\pi,c}_{t-} = 0\} = \inf\{t \geq 0; X^{\pi,0}_t = 0\}$. In addition, note that capital injection is only made when the surplus hits zero, so it is a regular stochastic control problem when the surplus is positive. Assume that the value function $V(x)$ of (2.4) is twice differentiable. Then, the dynamic programming principle (see Fleming and Soner (2006)) motivates that $V(x)$ satisfies the following HJB equation

$$\min_{\pi}(\mathcal{A}^\pi - \delta)V(x) = 0, \qquad x > 0, \tag{3.14}$$

with the boundary conditions (3.11) and (3.12). The following theorem shows us that if we can find a solution to the HJB equation (3.14) satisfying (3.11) and (3.12), then the solution must be the value function.

Theorem 3.1. *(Verification Theorem) If there exists a bounded twice differentiable function $W(x)$ such that $W'(0) < -1$ and $W(x)$ is a strictly convex decreasing solution to (3.14) with the boundary conditions (3.11) and (3.12), then we have*

$$W(x) = V(x), \qquad \text{for all } \ x \geq 0.$$

Proof. For any pair of admissible policies (π, c), the surplus process is described by $X^{\pi,c}_t = c$ if $X^{\pi,c}_{t-} = 0$ and

$$dX^{\pi,c}_t = \left[r_0 X^{\pi,c}_{t-} + (\alpha - r_0)\pi_t + \mu\right] dt + \sigma dB^X_t + \beta\pi_t dB^S_t, \quad \text{with } X^{\pi,c}_{0-} = x, \quad \text{if } X^{\pi,c}_{t-} > 0.$$

Applying the generalized Itô formula, we obtain

$$\begin{aligned} e^{-\delta t}W(X_t^{\pi,c}) &= W(x) + \int_0^t e^{-\delta s}(\mathcal{A}^\pi - \delta)W(X_{s-}^{\pi,c})ds + \sigma \int_0^t e^{-\delta s}W'(X_{s-}^{\pi,c})dB_s^X \qquad (3.15) \\ &\quad +\beta \int_0^t e^{-\delta s}\pi_s W'(X_{s-}^{\pi,c})dB_s^S + \sum_{0\le s\le t; X_s^{\pi,c}\neq X_{s-}^{\pi,c}} e^{-\delta s}[W(X_s^{\pi,c}) - W(X_{s-}^{\pi,c})] \\ &\ge W(x) + \sum_{0\le s\le t} e^{-\delta s}[W(c) - W(0+)]I_{\{X_{s-}^{\pi,c}=0\}} \\ &\quad +\sigma \int_0^t e^{-\delta s}W'(X_{s-}^{\pi,c})dB_s^X + \beta \int_0^t e^{-\delta s}\pi_s W'(X_{s-}^{\pi,c})dB_s^S \\ &\ge W(x) - \sum_{0\le s\le t} e^{-\delta s}(K+c)I_{\{X_{s-}^{\pi,c}=0\}} \\ &\quad +\sigma \int_0^t e^{-\delta s}W'(X_{s-}^{\pi,c})dB_s^X + \beta \int_0^t e^{-\delta s}\pi_s W'(X_{s-}^{\pi,c})dB_s^S, \qquad (3.16) \end{aligned}$$

where the first inequality follows from (3.14) and the second inequality is due to (3.12). Due to the boundedness of $W'(x)$ and (2.3), the last two terms in the last inequality are martingales. Taking the conditional expectation on both sides of the last inequality, we get

$$W(x) \le \mathbb{E}\left[\sum_{0\le s\le t} e^{-\delta s}(K+c)I_{\{X_{s-}^{\pi,c}=0\}} \mid X_{0-}^{\pi,c} = x\right] + \mathbb{E}\left[e^{-\delta t}W(X_t^{\pi,c})|X_{0-}^{\pi,c} = x\right]. \qquad (3.17)$$

Finally, letting $t \to \infty$ and taking infimum for all admissible policies, we have $W(x) \le V(x)$ for all $x \ge 0$.

On the other hand, if we take a pair of admissible policies (π^*, c^*) such that $(A^{\pi^*} - \delta)W(x) = 0$ for $x > 0$ and $c^* = \arg_{c>0} \min\{K + c + W(c)\}$, then the inequalities in (3.16) and (3.17) become equalities. So, we have $W(x) = V^{\pi^*,c^*}(x) \ge V(x)$ for all $x \ge 0$. Note that the convexity of $W(x)$ and $W'(0) < -1$ guarantees the existence of π^* and $c^* > 0$. □

4 Solutions to the optimization problem

In this section, we try to construct a solution to the HJB equation (3.14) with the boundary conditions (3.12) and (3.11) given that the conditions stated in the verification theorem are satisfied. If we can find a pair of admissible policies under which the performance function is the solution to (3.14), then, according to the verification theorem, this solution is the value function defined by (2.4). Also, it implies that the value function is indeed a twice differentiable function.

Assume that $W(x)$ is a convex candidate of the solution to (3.14) with the boundary conditions

(3.11) and (3.12). Then, $W(x)$ satisfies

$$\min_{\pi}\left\{\frac{1}{2}[\beta^2\pi^2+2\rho\beta\sigma\pi+\sigma^2]W''(x)+[r_0x+(\alpha-r_0)\pi+\mu]W'(x)\right\}-\delta W(x)=0. \tag{4.1}$$

Taking the derivative with respect to π on both sides of (4.1) and noting the convexity of $W(x)$, it follows that the minimum can be attained when

$$\pi(x)=-\frac{1}{\beta^2}(\alpha-r_0)\frac{W'(x)}{W''(x)}-\rho\frac{\sigma}{\beta}. \tag{4.2}$$

Let $m=(\alpha-r_0)/\beta$ be the market price of risk and $\tilde{\pi}(x)=\beta\pi(x)+\rho\sigma$. Then, (4.2) can be rewritten as

$$\frac{W''(x)}{W'(x)}=-\frac{m}{\tilde{\pi}(x)}. \tag{4.3}$$

After putting (4.2) or (4.3) into (4.1), we can cancel $\pi(x)$ out in (4.1). Hence, we have

$$\frac{1}{2}(1-\rho^2)\sigma^2W''(x)-\frac{1}{2}m^2\frac{(W'(x))^2}{W''(x)}+(r_0x+\mu-\rho\sigma m)\,W'(x)-\delta W(x)=0. \tag{4.4}$$

Due to the presence of r_0 in (4.4), it is impossible to find an explicit solution to the ODE. So, we opt for another way to tackle the problem. Instead of canceling $\pi(x)$ in (4.1), we can cancel $W''(x)$ out. It follows that

$$\left[r_0x+\mu-\rho\sigma m+\frac{1}{2}m\tilde{\pi}(x)-\frac{1}{2}(1-\rho^2)\sigma^2\frac{m}{\tilde{\pi}(x)}\right]W'(x)-\delta W(x)=0. \tag{4.5}$$

Then, taking derivative on both sides of (4.5) and using (4.3) again, we can show that $\tilde{\pi}(x)$ satisfies

$$\tilde{\pi}'(x)=\frac{(m-2\frac{r_0-\delta}{m})\tilde{\pi}^2(x)+2(r_0x+\mu-\rho\sigma m)\tilde{\pi}(x)-(1-\rho^2)\sigma^2m}{\tilde{\pi}^2(x)+(1-\rho^2)\sigma^2},\quad x\geq 0. \tag{4.6}$$

Again, we still cannot derive an explicit solution to the ODE (4.6) directly as we do not know the initial value $\tilde{\pi}(0)$. Fortunately, we are able to derive the limit of $\tilde{\pi}(x)$ as x tends to ∞ so that (4.6) can be used to solve for the optimal investment policy. Before doing so, we first try to obtain closed-form solutions in some special cases.

4.1 $|\rho|=1$

In this subsection, we study the case that the risky asset and the company's surplus are linearly correlated, that is, $|\rho|=1$. In this case, we have the following theorem.

Theorem 4.1. *For $|\rho|=1$, we have the following three results:*

(i) If $\rho = -1$, then the optimal investment policy $\pi_t^* = \sigma/\beta$ for all $t \geq 0$ and the surplus never hits zero. So, there is no capital injection, and hence $V(x) = 0$ for all $x > 0$.

(ii) If $\rho = 1$ and $\mu/\sigma \geq m$, then the optimal investment policy $\pi_t^* = -\sigma/\beta$ for all $t \geq 0$ and the surplus never hits zero. So, there is no capital injection, and hence $V(x) = 0$ for all $x > 0$.

(iii) If $\rho = 1$ and $\mu/\sigma < m$, then the optimal investment policy has the form

$$\pi_t^* = \begin{cases} \frac{\xi}{\beta}(\bar{x} - X_{t-}^{\pi^*,c^*}) - \frac{\sigma}{\beta}, & 0 \leq X_{t-}^{\pi^*,c^*} < \bar{x}, \\ -\frac{\sigma}{\beta}, & X_{t-}^{\pi^*,c^*} \geq \bar{x}, \end{cases} \tag{4.7}$$

where

$$\bar{x} = \frac{\sigma}{r_0}(m - \frac{\mu}{\sigma}), \qquad \xi = \frac{\sqrt{M^2 + 8r_0} - M}{2}, \qquad M = m - 2\frac{r_0 - \delta}{m}, \tag{4.8}$$

and the optimal capital injection amount $c^* > 0$ is uniquely determined by

$$\frac{\xi}{m+\xi}(\bar{x} - c)\left[\left(\frac{\bar{x}}{\bar{x} - c}\right)^{1+\frac{m}{\xi}} - 1\right] - c = K. \tag{4.9}$$

As a result, the value function is given by

$$V(x) = V^{\pi^*,c^*}(x) = \begin{cases} k_1(\bar{x} - x)^{1+\frac{m}{\xi}}, & 0 \leq x < \bar{x}, \\ 0, & x \geq \bar{x}, \end{cases} \tag{4.10}$$

where

$$k_1 = \frac{K + c^*}{\bar{x}^{1+\frac{m}{\xi}} - (\bar{x} - c^*)^{1+\frac{m}{\xi}}}. \tag{4.11}$$

Proof. For $\rho = -1$, the proof is straightforward. In fact, we can take the investment policy $\tilde{\pi}(x) = 0$, that is, $\pi(x) = \sigma/\beta$ for all $x \geq 0$. Then, the diffusion term in (2.2) can be completely canceled out and the drift coefficient turns out to be $r_0 x + \mu + \sigma m$, which is always positive for all $x \geq 0$. In this case, the surplus never hits zero. Therefore, the optimal investment policy is $\pi_t^* = \sigma/\beta$ and no capital injection policy is made. This implies that the value function $V(x) = 0$ for all $x \geq 0$, and hence (i) is proved.

If $\rho = 1$, although we can take the investment policy $\pi(x) = -\sigma/\beta$ for all $x \geq 0$ to cancel out the diffusion term in (2.2), the drift term becomes $r_0 x + \mu - \sigma m$ which is not always positive. If $\mu/\sigma \geq m$, the drift term is positive for all $x > 0$. So, the optimal investment policy is $\pi_t^* = -\sigma/\beta$ for all $t \geq 0$ and the value function $V(x) = 0$ for all $x \geq 0$. This gives (ii).

If $\rho = 1$ and $\mu/\sigma < m$, the optimal investment policy is not constant any more. Note that $\bar{x}$ is defined in (4.8). For $x \geq \bar{x}$, the drift term $r_0 x + \mu - \sigma m \geq 0$, so we still can take the optimal investment policy $\pi_t^* = -\sigma/\beta$ for all $t \geq 0$ so that the value function $V(x) = 0$ for all $x \geq \bar{x}$. For $0 \leq x < \bar{x}$, we need to construct a solution to the HJB equation (3.14) with the boundary conditions (3.12) and (3.11). Based on the analysis at the beginning of this section, the optimal investment policy should satisfy (4.6). Since $\rho = 1$, (4.6) reduces to

$$\tilde{\pi}'(x) = M + 2r_0 \frac{x - \bar{x}}{\tilde{\pi}(x)}, \qquad x \geq 0, \tag{4.12}$$

where M is given by (4.8). Without knowing the initial value $\tilde{\pi}(0)$, $\tilde{\pi}(x)$ cannot be solved directly from (4.12). However, with the condition $\tilde{\pi}(\bar{x}) = 0$, the ODE (4.12) has the following solution

$$\tilde{\pi}(x) = \xi(\bar{x} - x), \qquad 0 \leq x \leq \bar{x}, \tag{4.13}$$

and ξ is the root of

$$\xi^2 + M\xi - 2r_0 = 0. \tag{4.14}$$

In order to construct a convex solution to the HJB equation, we need to find a nonnegative $\tilde{\pi}(x)$ because of (4.3). Thus, we take ξ as the positive root of (4.14) which is given in (4.8).

Putting (4.13) back into (4.3) and noting the continuity of the value function at $\bar{x}$, that is, $W(\bar{x}) = 0$, we obtain

$$\begin{aligned} W'(x) &= -k_1 \left(1 + \frac{m}{\xi}\right) (\bar{x} - x)^{\frac{m}{\xi}}, && (4.15) \\ W(x) &= k_1 (\bar{x} - x)^{1 + \frac{m}{\xi}}, && (4.16) \end{aligned}$$

where $k_1 > 0$ is a constant. To determine the parameter k_1 and the optimal capital injection policy, we use the boundary condition (3.12). According to the convexity of the value function, it is necessary to find c^* such that

$$\begin{aligned} W'(c^*) &= -1, \\ W(c^*) &= W(0) - K - c^*. && (4.17) \end{aligned}$$

These together with (4.15) and (4.16) yield $c^* \in (0, \bar{x})$, which is the unique solution to the equation (4.9). In fact, let

$$h(c) = \frac{\xi}{m + \xi} (\bar{x} - c) \left[\left(\frac{\bar{x}}{\bar{x} - c} \right)^{1 + \frac{m}{\xi}} - 1 \right] - c.$$

Then, it is easy to check that $h(0) = 0$, $h(\bar{x}-) = +\infty$, and

$$h'(c) = \frac{m}{m + \xi} \left[\left(\frac{\bar{x}}{\bar{x} - c} \right)^{1 + \frac{m}{\xi}} - 1 \right] > 0, \quad \text{for} \quad 0 < c < \bar{x},$$

which imply that c^* uniquely exists. That is to say, it is optimal to inject c^* when the company's surplus hits zero. Then, combining (4.16) and (4.17), we obtain (4.11).

In view of the above analysis, we can find a solution $W(x)$ to the HJB equation with the boundary conditions (3.11) and (3.12). It is easy to check that this solution satisfies the conditions in Theorem 3.1. Also, it can be shown that both the investment policy π^* and the capital injection policy c^* defined by (4.7) and (4.24), respectively, are admissible policies, and that $W(x) = V^{\pi^*,c^*}(x)$. Then, (4.10) follows from the the verification theorem. This completes the proof of (iii). □

4.2 $|\rho| < 1$

Recall the ODE (4.6) satisfied by the optimal investment policy. It is not possible to uniquely determine a solution to the above equation without any boundary condition. Before we further investigate the optimal investment policy, we now present some properties of the solutions to (4.6).

Let $\eta_+ > 0$ and $\eta_- < 0$ be the two roots of the equation

$$\left(m + \frac{2\delta}{m}\right) x^2 + 2(\mu - \rho\sigma m)x - (1-\rho^2)\sigma^2 m = 0.$$

Lemma 4.1. *If* $\lim_{x\to\infty} f(x)$ *is finite and* $\lim_{x\to\infty} f'(x)$ *exists (the existence of the limit here means that it may be either a finite real number,* ∞*, or* $-\infty$*), then* $\lim_{x\to\infty} f'(x) = 0$.

Proof. The proof of Lemma 4.1 is straightforward. □

Lemma 4.2. *Let* $\tilde{\pi}(x)$ *be a solution to the ODE (4.6). Then, the following two statements hold:*

(i) If $r_0 = 0$*, either* $\lim_{x\to\infty} \tilde{\pi}(x) = \infty$, $\tilde{\pi}(x) = \eta_+$ *for all* $x \geq 0$*, or* $\lim_{x\to\infty} \tilde{\pi}(x) = \eta_-$ *holds;*

(ii) If $r_0 > 0$*, either* $\lim_{x\to\infty} \tilde{\pi}(x) = \infty$, $\lim_{x\to\infty} \tilde{\pi}(x) = 0$*, or* $\lim_{x\to\infty} \tilde{\pi}(x) = -\infty$ *holds. Furthermore, if* $\lim_{x\to\infty} \tilde{\pi}(x) \neq -\infty$*, we have* $\tilde{\pi}(x) > 0$ *for all* $x \geq 0$.

Proof. Here, we only present the proof for $m - 2(r_0 - \delta)/m > 0$. For $m - 2(r_0 - \delta)/m \leq 0$, the proof is similar.

We first prove (i) with $r_0 = 0$. Note that $\eta_+ > 0$ and $\eta_- < 0$ are the two zeros of the numerator in (4.6). Thus, it is clear that both $\tilde{\pi}(x) = \eta_+$ and $\tilde{\pi}(x) = \eta_-$ are two solutions to the ODE (4.6). If $\tilde{\pi}(0) > \eta_+$, then we have $\tilde{\pi}'(0) > 0$, which implies that $\tilde{\pi}(x) > \eta_+$ and $\tilde{\pi}'(x) > 0$ for all $x \geq 0$. If there exists $\eta_+ < \Delta < \infty$ such that $\lim_{x\to\infty} \tilde{\pi}(x) = \Delta$, then, by taking $x \to \infty$ on both sides of (4.6), one can show that $\lim_{x\to\infty} \tilde{\pi}'(x)$ exists and is positive. Hence, it follows from Lemma 4.1 that

$\lim_{x\to\infty}\tilde{\pi}'(x)=0$, which leads to a contradiction. Similarly, we can conclude that, for $\tilde{\pi}(0)<\eta_+$, $\lim_{x\to\infty}\tilde{\pi}(x)=\eta_-$.

For (ii) with $r_0>0$, we first show that $\lim_{x\to\infty}\tilde{\pi}(x)$ cannot be a nonzero finite constant. If there exists a finite constant $\Delta\neq 0$ such that $\lim_{x\to\infty}\tilde{\pi}(x)=\Delta$, then, by taking $x\to\infty$ on both sides of (4.6), we have $\lim_{x\to\infty}\tilde{\pi}'(x)=\infty$ or $-\infty$, which again shows a contradiction because of Lemma 4.1.

Note that the numerator in (4.6) is a quadratic polynomial with respect to $\tilde{\pi}(x)$. Letting the numerator be zero, we obtain the positive solution

$$\hat{\pi}_+(x)=\frac{(1-\rho^2)\sigma^2 m}{\sqrt{(r_0x+\mu-\rho\sigma m)^2+(1-\rho^2)\sigma^2[m^2-2(r_0-\delta)]}+r_0x+\mu-\rho\sigma m}$$

and the negative solution

$$\hat{\pi}_-(x)=-\frac{\sqrt{(r_0x+\mu-\rho\sigma m)^2+(1-\rho^2)\sigma^2[m^2-2(r_0-\delta)]}+r_0x+\mu-\rho\sigma m}{m-2(r_0-\delta)/m}.$$

It is easy to see that both $\hat{\pi}_+(x)$ and $\hat{\pi}_-(x)$ are strictly decreasing, $\lim_{x\to\infty}\hat{\pi}_+(x)=0$, and $\lim_{x\to\infty}\hat{\pi}_-(x)=-\infty$. To complete the proof of (ii), we need to discuss the following four cases:

(1) If there exists $x'\geq 0$ such that $\tilde{\pi}(x')\geq\hat{\pi}_+(x')$, then this together with (4.6) gives $\tilde{\pi}'(x')\geq 0$. Since $\hat{\pi}_+(x)$ is strictly decreasing with respect to x, it follows that $\tilde{\pi}(x)>\hat{\pi}_+(x)$ for all $x>x'$. Again, due to (4.6), $\tilde{\pi}'(x)>0$ for all $x\geq x'$. Thus, we can conclude that $\lim_{x\to\infty}\tilde{\pi}(x)=\infty$ as the limit cannot be a nonzero finite constant.

(2) If there exists $x'\geq 0$ such that $\hat{\pi}_-(x')\leq\tilde{\pi}(x')\leq 0$, then (4.6) yields $\tilde{\pi}'(x)\leq 0$ for all $x>x'$. So, we have $\lim_{x\to\infty}\tilde{\pi}(x)=-\infty$ as the limit cannot be a nonzero finite constant.

(3) If $0<\tilde{\pi}(x)<\hat{\pi}_+(x)$ for all x, then $0\leq\lim_{x\to\infty}\tilde{\pi}(x)\leq\lim_{x\to\infty}\hat{\pi}_+(x)=0$.

(4) If $\tilde{\pi}(x)<\hat{\pi}_-(x)$ for all $x\geq 0$, then (4.6) yields $\tilde{\pi}'(x)>0$ for all $x\geq 0$. Note that $\hat{\pi}_-(x)$ is strictly decreasing and $\lim_{x\to\infty}\hat{\pi}_-(x)=-\infty$. Hence, this case is impossible.

□

Despite the fact that we do not have the initial value of $\tilde{\pi}(x)$, we can derive the limit of the optimal investment strategy $\tilde{\pi}(x)$ by using the properties of the value function and the ODE satisfied by the optimal investment policy. In the following theorem, we give the limiting value of the optimal investment policy which is the key step to solve the stochastic optimization problem.

Theorem 4.2. *For $|\rho| < 1$, if $W(x)$ satisfies all the conditions of the verification theorem and the optimal investment policy $\tilde{\pi}(x)$ is differentiable, then we have:*

(i) The optimal investment policy $\tilde{\pi}(x) > 0$ for $x \geq 0$;

(ii) If $r_0 = 0$, then the optimal investment policy $\tilde{\pi}(x) = \eta_+$ for all $x \geq 0$;

(iii) If $r_0 > 0$, then the optimal investment policy satisfies that $\lim_{x\to\infty} \tilde{\pi}(x) = 0$. If $\lim_{x\to\infty} \tilde{\pi}'(x)$ exists, then $\lim_{x\to\infty} \tilde{\pi}'(x) = 0$ and $\lim_{x\to\infty} x\tilde{\pi}(x) = m(1-\rho^2)\sigma^2/2r_0$.

Proof. Note that $W(x)$, the candidate solution to the value function, should be a decreasing function. It follows from (4.3) that $\tilde{\pi}(x) \geq 0$ for all $x \geq 0$ when $W(x)$ is convex. In addition, with $|\rho| < 1$, there does not exist an investment policy by which the diffusion term can be completely canceled out, because the diffusion coefficient is always larger than or equal to $\sqrt{1-\rho^2}\sigma > 0$. From the time change of martingales, we see that for any initial surplus $x \geq 0$, there is a positive probability that the surplus hits zero and hence capital injection is necessary. Thus, $W(x) > 0$ for all $x \geq 0$. From the boundary condition $\lim_{x\to\infty} W(x) = 0$ and the fact $W(x)$ is a decreasing function, it can be shown that $W'(x) < 0$ for all $x \geq 0$, and that $\lim_{x\to\infty} W'(x) = 0$. Thus, $\tilde{\pi}(x) > 0$ for all $x \geq 0$ due to (4.3). So, (i) is proved.

In order to prove (ii) and (iii), we first show that $\lim_{x\to\infty} \tilde{\pi}(x) \neq \infty$. Now assume that $\lim_{x\to\infty} \tilde{\pi}(x) = \infty$. Taking limits on both sides of (4.3), we get

$$\lim_{x\to\infty} \frac{W''(x)}{W'(x)} = -\lim_{x\to\infty} \frac{m}{\tilde{\pi}(x)} = 0, \text{ and then } \lim_{x\to\infty} \frac{W'(x)}{W''(x)} = -\infty.$$

Then, by l'Hopital's rule, we have

$$\lim_{x\to\infty} \frac{W(x)}{W'(x)} = \lim_{x\to\infty} \frac{W'(x)}{W''(x)} = -\infty.$$

Recall the ODE (4.4) and note that $W'(x) < 0$ for all $x \geq 0$. Dividing by $W'(x)$ on both sides of (4.4) and taking the limit $x \to \infty$, we obtain

$$\lim_{x\to\infty} \left\{ \frac{1}{2}(1-\rho^2)\sigma^2 \frac{W''(x)}{W'(x)} - \frac{1}{2}m^2 \frac{W'(x)}{W''(x)} + [r_0 x + \mu - \rho\sigma m] - \delta \frac{W(x)}{W'(x)} \right\} = +\infty \neq 0,$$

which shows a contradiction. Therefore, we conclude that $\lim_{x\to\infty} \tilde{\pi}(x) \neq \infty$.

Then, according to Lemma 4.2, we know that $\tilde{\pi}(x) = \eta_+$ for all $x \geq 0$ when $r_0 = 0$ and $\lim_{x\to\infty} \tilde{\pi}(x) = 0$ when $r_0 > 0$. Furthermore, for $r_0 > 0$, we have $\lim_{x\to\infty} \tilde{\pi}'(x) = 0$. Taking limits on both sides of (4.6), we obtain $\lim_{x\to\infty} x\tilde{\pi}(x) = m(1-\rho^2)\sigma^2/2r_0$. Therefore, both (ii) and (iii) are proved. □

In the above theorem, we obtain a candidate for the optimal investment policy when $r_0 = 0$, and the limiting value of the optimal investment policy when $r_0 > 0$. Then, in the following subsections, we shall solve the value functions and the optimal policies for the two cases.

4.2.1 $r_0 = 0$

Theorem 4.3. *If $r_0 = 0$, then the optimal investment policy is given by*

$$\pi_t^* = \eta_+, \qquad \text{for all } \; t \geq 0,$$

and the optimal capital injection amount $c^* > 0$ is uniquely determined by the equation

$$e^{\frac{m}{\eta_+}c} = \frac{m}{\eta_+}(c+K)+1. \tag{4.18}$$

As a result, the value function has the form

$$V(x) = \frac{K+c^*}{1-e^{-\frac{m}{\eta_+}c^*}}e^{-\frac{m}{\eta_+}x}, \quad x \geq 0. \tag{4.19}$$

Proof. For $r_0 = 0$, it follows from Theorem 4.2 that $\tilde{\pi}(x) = \eta_+$ for all $x \geq 0$. Then, from (4.3), we have

$$\begin{aligned} W'(x) &= -k_2\frac{m}{\eta_+}e^{-\frac{m}{\eta_+}x}, \\ W(x) &= k_2 e^{-\frac{m}{\eta_+}x}. \end{aligned}$$

To determine the parameter k_2 and the optimal capital injection policy, we use the boundary condition (3.12). Because of the convexity of $W(x)$, it is sufficient to find c^* such that

$$\begin{aligned} W'(c^*) &= -1, \\ W(c^*) &= W(0) - K - c^*. \end{aligned}$$

So, $c^* > 0$ is determined by the equation (4.18). It is easy to check that c^* is unique. By solving $W(0) = W(c^*) + K + c^*$, we obtain $k_2 = (K+c^*)/(1-e^{-m/\eta_+ c^*})$. By direct calculation, one can verify that $W(x)$ obtained here satisfies all the conditions of the verification theorem. Hence, the theorem follows from the verification theorem. □

4.2.2 $r_0 > 0$

As is shown in Theorem 4.2, we need to find a positive solution to (4.6) such that $\lim_{x\to\infty}\tilde{\pi}(x) = 0$ when $r_0 > 0$. However, an explicit solution is difficult to obtain except for the special case of

$\delta = 0$ (see Remark 4.1 below). In the following theorem, we first give a general solution in series form.

Theorem 4.4. *Given x_0 defined in (4.24) below, the ODE (4.6) together with the terminal condition* $\lim_{x\to\infty} \tilde{\pi}(x) = 0$ *determines a unique solution on* $[0, \infty)$ *such that*

$$\tilde{\pi}(x) = \sum_{k=1}^{\infty} a_k x^{-k}, \quad \text{for } x > x_0, \tag{4.20}$$

where the coefficients a_k for $k = 1, 2, \cdots$ are calculated recursively by (4.21)-(4.23) given below.

Proof. Inspired by (iii) of Theorem 4.2, we guess that the optimal investment policy $\tilde{\pi}(x)$ can be written as a series function (4.20). Clearly, we have

$$\tilde{\pi}^2(x) = \sum_{k=2}^{\infty} b_k x^{-k}, \quad \text{with} \quad b_k = \sum_{1\le i\le k-1} a_i a_{k-i}, k \ge 2.$$

Assume that, for large enough x, the series function (4.20) is uniformly convergent. Then, it follows that

$$\tilde{\pi}'(x) = \sum_{k=2}^{\infty} a_{k-1}(1-k)x^{-k}.$$

Putting (4.20), $\tilde{\pi}^2(x)$ and $\tilde{\pi}'(x)$ into (4.6), we have

$$\begin{aligned}\sum_{k=4}^{\infty}\left(\sum_{2\le i\le k-2} a_{i-1}(1-i)b_{k-i}\right)x^{-k} &= \sum_{k=2}^{\infty}\left[\left(m - \frac{2(r_0-\delta)}{m}\right)b_k + 2(r_0 a_{k+1} + (\mu - \rho\sigma m)a_k)\right]x^{-k} \\ &\quad +2[r_0 a_2 + (\mu-\rho\sigma m)a_1]x^{-1} + 2r_0 a_1 - (1-\rho^2)\sigma^2 m.\end{aligned}$$

Comparing all the coefficients of x^{-k} for $k = 1, 2, \cdots$, we obtain

$$a_1 = \frac{(1-\rho^2)\sigma^2}{2r_0}, \tag{4.21}$$

$$a_2 = -\frac{\mu-\rho\sigma m}{r_0}a_1, \tag{4.22}$$

and, for $k = 2, 3, \cdots$,

$$a_{k+1} = -\frac{\mu-\rho\sigma m}{r_0}a_k - \frac{1}{2r_0}\left(m - \frac{2(r_0-\delta)}{m}\right)b_k + \frac{1}{2r_0}\left(\sum_{2\le i\le k-2} a_{i-1}(1-i)b_{k-i}\right). \tag{4.23}$$

Define a constant x_0 as

$$x_0 = \max\left\{\left|\frac{\mu-\rho\sigma m}{r_0}\right|, a_1\left|\frac{\mu-\rho\sigma m}{r_0}\right|, \frac{1}{2r_0}\left|m - \frac{2(r_0-\delta)}{m}\right|, \frac{1}{2r_0}\right\}. \tag{4.24}$$

Then, by the recursion formulas, it follows that $|a_k| \leq x_0^k$ for large enough k. Also, note that $\sum_{k=1}^{\infty}(x_0/x)^k$ is uniformly convergent on $[x_1, \infty)$ for any $x_1 > x_0$. Thus, the series function $\sum_{k=1}^{\infty} a_k x^{-k}$ is uniformly convergent on $[x_1, \infty)$ for any $x_1 > x_0$.

Note that the coefficients a_k are uniquely determined for $x > x_0$. Then, the solution is unique on (x_0, ∞). Also, note that the right hand side of (4.6) is not singular. By the continuation of solutions to the ODE, it follows that the solution is unique on $[0, \infty)$. This completes the proof. □

Theorem 4.3 and Theorem 4.4 tell us that the ODE (4.6) together with the terminal condition $\lim_{x\to\infty} \tilde{\pi}(x) = 0$ determines a unique positive solution, which is a candidate for the optimal investment policy. Once the optimal investment policy $\tilde{\pi}(x)$ is determined, the value function can be derived using (4.3).

Theorem 4.5. *If $r_0 > 0$, the optimal investment policy is given by*

$$\pi_t^* = \tilde{\pi}\left(X_{t-}^{\pi^*, c^*}\right),$$

and the optimal capital injection amount $c^* > 0$ is uniquely given by the solution to the equation

$$\int_0^c \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)} ds\right) dy \cdot \exp\left(\int_0^c \frac{m}{\tilde{\pi}(s)} ds\right) - c = K. \tag{4.25}$$

As a result, the value function has the form

$$V(x) = k \int_x^{\infty} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)} ds\right) dy, \qquad x \geq 0,$$

where k is given by

$$k = \frac{K + c^*}{\int_0^{c^*} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)} ds\right) dy} = \exp\left(\int_0^{c^*} \frac{m}{\tilde{\pi}(s)} ds\right). \tag{4.26}$$

Proof. It follows from (4.3) that

$$\begin{aligned} W'(x) &= -k \exp\left(-\int_0^x \frac{m}{\tilde{\pi}(s)} ds\right), \\ W(x) &= W(0) - k \int_0^x \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)} ds\right) dy. \end{aligned}$$

These together with the boundary condition $W(\infty) = 0$ yield

$$W(x) = k \int_x^{\infty} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)} ds\right) dy,$$

where k needs to be determined later. Similarly, because of the convexity of $W(x)$, it is sufficient to find c^* such that

$$\begin{aligned} W'(c^*) &= -1, \\ W(c^*) &= W(0) - K - c^*. \end{aligned}$$

From the first equation, we obtain $k = \exp\left(\int_0^{c^*} m/\tilde{\pi}(s)ds\right)$. In order to determine c^* from the second equation, we define

$$g(c) = \int_0^c \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right) dy \cdot \exp\left(\int_0^c \frac{m}{\tilde{\pi}(s)}ds\right) - c, \quad c \geq 0.$$

Note that $\tilde{\pi}(x) > 0$ for all $x \geq 0$. Then, it is easy to check that $g(0) = 0$ and $g(c) \geq 0$. In addition, since $\tilde{\pi}(x) \sim a_1/x$ for $x \to \infty$, we have

$$\int_0^c \frac{m}{\tilde{\pi}(s)}ds \sim \frac{m}{2a_1}c^2, \quad \text{as} \ \ c \to \infty,$$

which implies that

$$\int_0^\infty \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right) dy < \infty,$$

and hence $\lim_{c\to\infty} g(c) = \infty$. Also, we have

$$g'(c) = (g(c) + c)\frac{m}{\tilde{\pi}(c)} > 0.$$

Therefore, we conclude that c^* can be uniquely determined from (4.25), and k is given by (4.26).

In what follows we verify that $\tilde{\pi}(x)$ given by Theorem 4.4 and $W(x)$ obtained above satisfy the boundary conditions (3.11) and (3.12), and the HJB equation (4.1).

Firstly, it is obvious that $\lim_{x\to\infty} W(x) = 0$. For the boundary condition (3.12), it follows by

$$\begin{aligned} W(c^*) + K + c^* &= W(0) - k\int_0^{c^*} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right) dy + K + c^* \\ &= W(0) - \frac{K + c^*}{\int_0^{c^*} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right) dy}\int_0^{c^*} \exp\left(-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right) dy + K + c^* \\ &= W(0). \end{aligned}$$

Now we verify the HJB equation (4.1). Note that

$$\begin{aligned} W'(x) &= -k\exp\left(-\int_0^x \frac{m}{\tilde{\pi}(s)}ds\right), \\ W''(x) &= W'(x) = k\exp\left(-\int_0^x \frac{m}{\tilde{\pi}(s)}ds\right)\frac{m}{\tilde{\pi}(x)}, \end{aligned}$$

which imply that

$$\frac{W''(x)}{W'(x)} = -\frac{m}{\tilde{\pi}(x)}.$$

Thus, we just need to verify that $W(x)$ and $\tilde{\pi}(x)$ satisfy (4.5), that is

$$L(x) \triangleq \frac{m\tilde{\pi}^2(x) + 2(r_0x + \mu - \rho\sigma m)\tilde{\pi}(x) - (1-\rho^2)\sigma^2 m}{\tilde{\pi}(x)} = 2\delta\frac{W(x)}{W'(x)}$$
$$= -2\delta\int_x^\infty \exp\left\{-\int_0^y \frac{m}{\tilde{\pi}(s)}ds\right\}dy\exp\left\{\int_0^x \frac{m}{\tilde{\pi}(s)}ds\right\} \triangleq R(x).$$

Then, by l'Hopital's rule, it follows that

$$\lim_{x\to\infty} R(x) = -2\delta\lim_{x\to\infty}\frac{\tilde{\pi}(x)}{m} = 0,$$

and $R(x)$ satisfies the ODE

$$R'(x) = 2\delta + R(x)\frac{m}{\tilde{\pi}(x)}.$$

In addition,

$$\begin{aligned}
\lim_{x\to\infty} L(x) &= \lim_{x\to\infty}\frac{2r_0x\tilde{\pi}(x) - (1-\rho^2)\sigma^2 m}{\tilde{\pi}(x)} + 2(\mu - \rho\sigma m)\\
&= \lim_{x\to\infty}\frac{(2r_0a_1 - (1-\rho^2)\sigma^2 m) + 2r_0\sum_{k=1}^\infty a_{k+1}x^{-k}}{\sum_{k=1}^\infty a_kx^{-k}} + 2(\mu - \rho\sigma m)\\
&= 2r_0\frac{a_2}{a_1} + 2(\mu - \rho\sigma m)\\
&= 0,
\end{aligned}$$

where the last two steps are due to (4.21) and (4.22). Also, $L(x)$ satisfies the ODE:

$$\begin{aligned}
L'(x) &= \frac{2[m\tilde{\pi}^2(x) + (r_0x + \mu - \rho\sigma m)\tilde{\pi}(x)]\tilde{\pi}'(x) + 2r_0\tilde{\pi}^2(x)}{\tilde{\pi}^2(x)}\\
&\quad -[m\tilde{\pi}^2(x) + 2(r_0x + \mu - \rho\sigma m)\tilde{\pi}(x) - (1-\rho^2)\sigma^2 m]\frac{\tilde{\pi}'(x)}{\tilde{\pi}^2(x)}\\
&= 2r_0 + m[\tilde{\pi}^2(x) + (1-\rho^2)\sigma^2]\frac{\tilde{\pi}'(x)}{\tilde{\pi}^2(x)}\\
&= 2\delta + L(x)\frac{m}{\tilde{\pi}(x)},
\end{aligned}$$

where (4.6) is used in the last step. In conclusion, we have $\lim_{x\to\infty} L(x) - R(x) = 0$ and $L(x) - R(x)$ satisfying the ODE

$$(L(x) - R(x))' = (L(x) - R(x))\frac{m}{\tilde{\pi}(x)},\ x \geq 0.$$

Since $\tilde{\pi}(x) > 0$ for all x, the unique solution is $L(x) - R(x) = 0$, that is

$$L(x) = R(x),\ x \geq 0.$$

Finally, a direct application of the verification theorem gives the theorem. □

Though we cannot in general find the optimal investment policy in closed form, we can derive an explicit expression for the optimal investment policy in the case of the discount rate $\delta = 0$. In fact, in the following numerical examples, we can see that the explicit expression for the optimal investment policy in the case of $\delta = 0$ is close to that in the case of $\delta \neq 0$, especially for an insurance company with a large initial surplus.

Remark 4.1. (Optimal investment policy with $\delta = 0$)

In the case of $\delta = 0$, the value function is obtained by minimizing the expected value of the total capital injection cost without discount. In this case, the optimal investment policy can be obtained in closed form. In fact, it follows from (4.5) that

$$\left[r_0 x + \mu - \rho\sigma m + \frac{1}{2} m\tilde{\pi}(x) - \frac{1}{2}(1-\rho^2)\sigma^2 \frac{m}{\tilde{\pi}(x)}\right] W'(x) = 0,$$

which implies that the optimal investment policy is the positive root of the following equation

$$m\tilde{\pi}^2(x) + 2(r_0 x + \mu - \rho\sigma m)\tilde{\pi}(x) - (1-\rho^2)\sigma^2 m = 0. \tag{4.27}$$

Hence, we have

$$\tilde{\pi}(x) = \sqrt{\left(\frac{r_0}{m}x + \frac{\mu}{m} - \rho\sigma\right)^2 + (1-\rho^2)\sigma^2} - \left(\frac{r_0}{m}x + \frac{\mu}{m} - \rho\sigma\right). \tag{4.28}$$

It is easy to verify that the solution to (4.27) is also the solution to (4.6), and (4.28) satisfies the condition that $\lim_{x\to\infty} \tilde{\pi}(x) = 0$. Note that the optimal investment policy (4.28) is the same as the one that minimizes the probability of ruin in the diffusion risk model (see Browne (1995) for details).

5 Numerical examples

In the previous section, Theorem 4.4 gives an explicit expression for the optimal investment policy for $x > x_0$, but we do not know much about the optimal investment policy when $0 \le x \le x_0$. In this section, we propose a numerical method to solve the optimal investment policy, which can help us understand more about the optimal investment policy for $x > 0$.

Recall again the ODE (4.6) satisfied by the optimal investment policy. In order to facilitate the computation, we perform a change of variable. Let $\omega(x) = \tilde{\pi}(1/x)$. Then, we have $\tilde{\pi}'(1/x) = -x^2\omega(x)$, and (4.6) can be translated to

$$\omega'(x) = -\frac{(m - 2\frac{r_0-\delta}{m})\omega^2(x) + 2(\frac{r_0}{x} + (\mu - \rho\sigma m))\omega(x) - (1-\rho^2)\sigma^2 m}{x^2(\omega^2(x) + (1-\rho^2)\sigma^2)}, \quad x > 0, \tag{5.1}$$

with $\omega(0) = 0$. Note that (5.1) is a singular ODE at $x = 0$. In order to avoid the singularity point $x = 0$, we can employ the backward Euler method. Given a small $h > 0$, choose a partition $x_n = nh,\ n = 1, 2, \cdots,$ for the interval $[0, \infty)$ and define $\omega_n = \omega(x_n)$. Integrating from x_n to x_{n+1} on both sides of (5.1), we have

$$\begin{aligned}\omega_{n+1} &= \omega_n - \int_{x_n}^{x_{n+1}} \frac{(m - 2\frac{r_0-\delta}{m})\omega^2(x) + 2(\frac{r_0}{x} + \mu - \rho\sigma m)\omega(x) - (1-\rho^2)\sigma^2 m}{x^2(\omega^2(x) + (1-\rho^2)\sigma^2)} dx \\ &\approx \omega_n - \frac{(m - 2\frac{r_0-\delta}{m})\omega_{n+1}^2 + 2(\frac{r_0}{x_{n+1}} + \mu - \rho\sigma m)\omega_{n+1} - (1-\rho^2)\sigma^2 m}{x_{n+1}^2(\omega_{n+1}^2 + (1-\rho^2)\sigma^2)} h,\end{aligned}$$

where the last equation is obtained by approximation at the right point. If ω_n is given, then ω_{n+1} can be approximated by solving the following cubic equation

$$\begin{aligned}x_{n+1}^2\omega_{n+1}^3 &+ \left[\left(m - 2\frac{r_0 - \delta}{m}\right) h + x_{n+1}^2\omega_n\right]\omega_{n+1}^2 \\ &+ 2\left(\frac{r_0}{x_{n+1}} + \mu - \rho\sigma m\right) h\omega_{n+1} - (1-\rho^2)\sigma^2\left(mh + x_{n+1}^2\omega_n\right) = 0.\end{aligned}$$

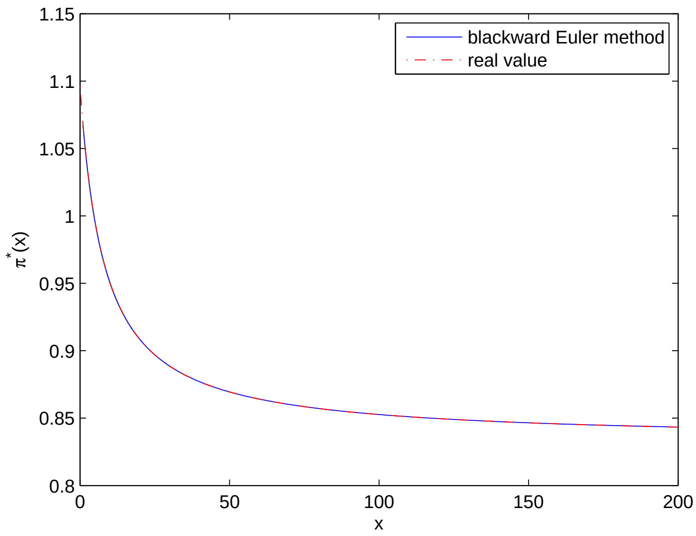
Note that the constant term of the above equation is negative. So, there exists at least one real positive root for this cubic equation. We take ω_{n+1} as the minimal positive real root and do the recursion. If the step width h is small enough, the solution to the IVP of (5.1) can be solved numerically. In the following, we carry out a few numerical examples to assess the performance of the backward Euler method.

In this section, the values of the model parameters are set as follows: the risk-free interest rate $r_0 = 0.05$, the expected return of the risky asset $\alpha = 0.15$ with a constant volatility $\beta = 0.3$, the drift and volatility of the portfolio's surplus $\mu = 0.3$ and $\sigma = 0.5$, the correlation coefficient of the risky asset and the portfolio's surplus $\rho = -0.5$, and the ratio of the fixed cost to the proportional cost for capital injection $K/l = 2$.

Based on these values, we first compare the approximated optimal investment policy obtained using the backward Euler method with the exact one obtained from (4.28) in the case of $\delta = 0$, by plotting the optimal investment policy $\pi^*(x)$ against the initial surplus x. Figure 1 shows that the two curves are very close to each other.

Given the parameter values, we apply the backward Euler method to find approximate solutions to the IVP of (5.1) in the following examples. In particular, the numerical examples show how the optimal policies change as one of parameter values changes and the other parameter values are kept fixed.

Figure 1: The plot of optimal investment policies with $\delta = 0$ against initial surplus



Example 1 (Effect of the discount rate δ). In Figure 2, the two panels present the effect of the discount rate on the optimal investment policy and the optimal amount of capital injection, respectively. In practice, the value of the discount rate is usually not larger than 30%. However, in order to display comparisons clearly, we take $\delta = 0, 0.1, 1$ in the upper panel of Figure 2. We see that the effect of the discount rate on the optimal investment policy is quite small and that the three sets of optimal investment policies are very close to each other. This phenomenon, noticed for our specific model parameters, suggests that the optimal investment policy with $\delta = 0$ given by

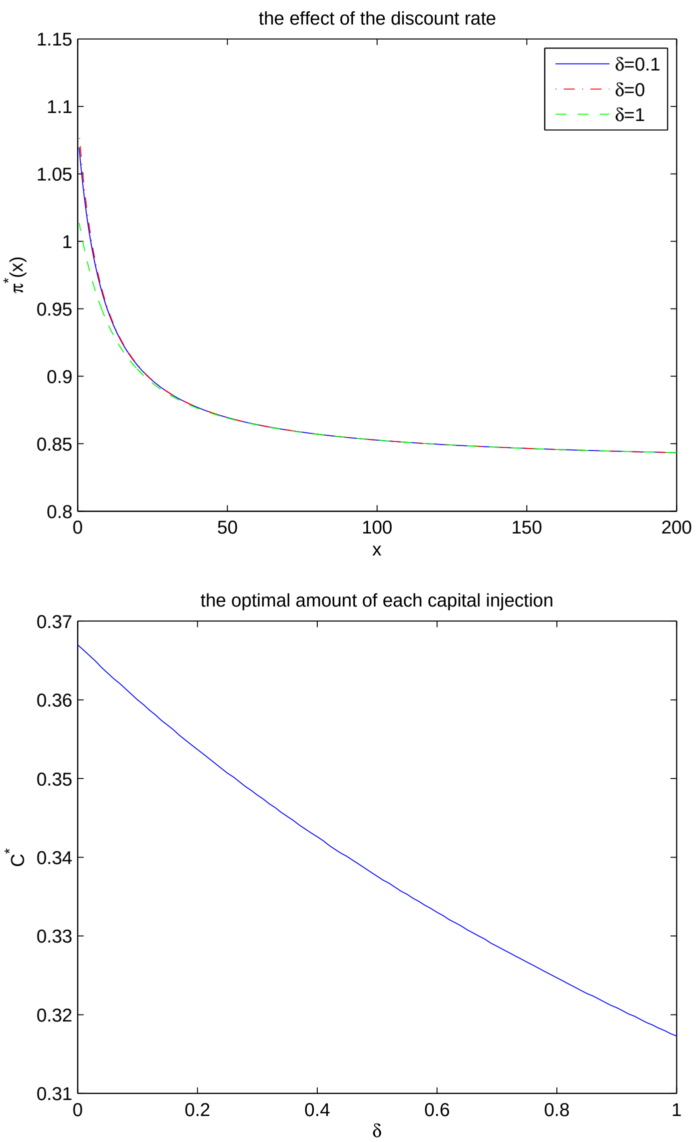
$$\pi^*(x) = \frac{1}{\beta}\left[\sqrt{\left(\frac{r_0}{m}x + \frac{\mu}{m} - \rho\sigma\right)^2 + (1-\rho^2)\sigma^2} - \left(\frac{r_0}{m}x + \frac{\mu}{m}\right)\right], \quad x \geq 0,$$

might serve as a crude estimate for that with $\delta > 0$. In the lower panel of Figure 2, the optimal amount c^* decreases as the discount rate increases. This may be due to the fact that, in order to minimize the discounted capital injection cost, it is better to delay part of each capital injection if the discounted rate is large. With capital injection, the surplus eventually tends to infinity. Also, there are at most a finite number of capital injection and most of them happen at early times. □

In the following examples, we set $\delta = 0.1$.

Example 2 (Effect of the correlation coefficient ρ). In the upper panel of Figure 3, the optimal investment amount in the risk-free asset decreases with the correlation coefficient ρ. In the

Figure 2: Effect of the discount rate δ



financial market, the best way to diversify the risk is to invest in a risky asset which is negatively correlated to one's own business. If there is only a positively correlated risky asset in the market, one can sell short this asset to diversify the underlying risk of the business. The effect of the correlation coefficient on the optimal amount of each capital injection looks interesting in the lower panel of Figure 3. The shape of the optimal amount is bell shaped. The shape is not completely symmetrical at $\rho = 0$, and the peak is attained when ρ is slightly larger than 0. If $|\rho| = 1$, the risk can be completely diversified and the surplus never hits zero, so the optimal amount of capital injection is zero. □

Example 3 (Effect of the risk-free interest rate r_0). In the lower panel of Figure 4, we see that the optimal investment policy decreases with the risk-free interest rate r_0. If r_0 increases, the investor would like to reduce the amount of risky investment and invest more in the risk-free asset. The lower panel of Figure 4 reveals the fact that the optimal amount of each capital injection decreases as the risk-free interest rate increases. If r_0 becomes larger, the probability of the surplus hitting zero gets smaller. In this situation, in order to minimize the transaction costs, the amount of each capital injection tends to be small. □

Example 4 (Effect of the surplus volatility σ). As we can see in the upper panel of Figure 5, the optimal investment amount in the risky asset increases with the surplus volatility σ. It indicates that the larger the underlying risk of the insurance portfolio, the larger the amount invested in the risky asset. Again, this phenomenon is due to risk diversification. In the lower panel of Figure 5, we see that the optimal amount of capital injection increases with the surplus volatility σ since the probability of the surplus of hitting zero is small (large) when σ is small (large). □

Example 5 (Effect of transaction costs). In Figure 6, we see that the optimal amount of capital injection increases with the ratio of the fixed cost to the proportional cost K/l. For a fixed l, the investor would like to inject more when the surplus hits zero if the fixed cost is large. For a fixed K, the investor would like to inject less if the proportional cost is large since a larger injection leads to a larger transaction cost. Note that the transaction costs do not affect the optimal investment policy. □

Figure 3: Effect of the correlation coefficient ρ

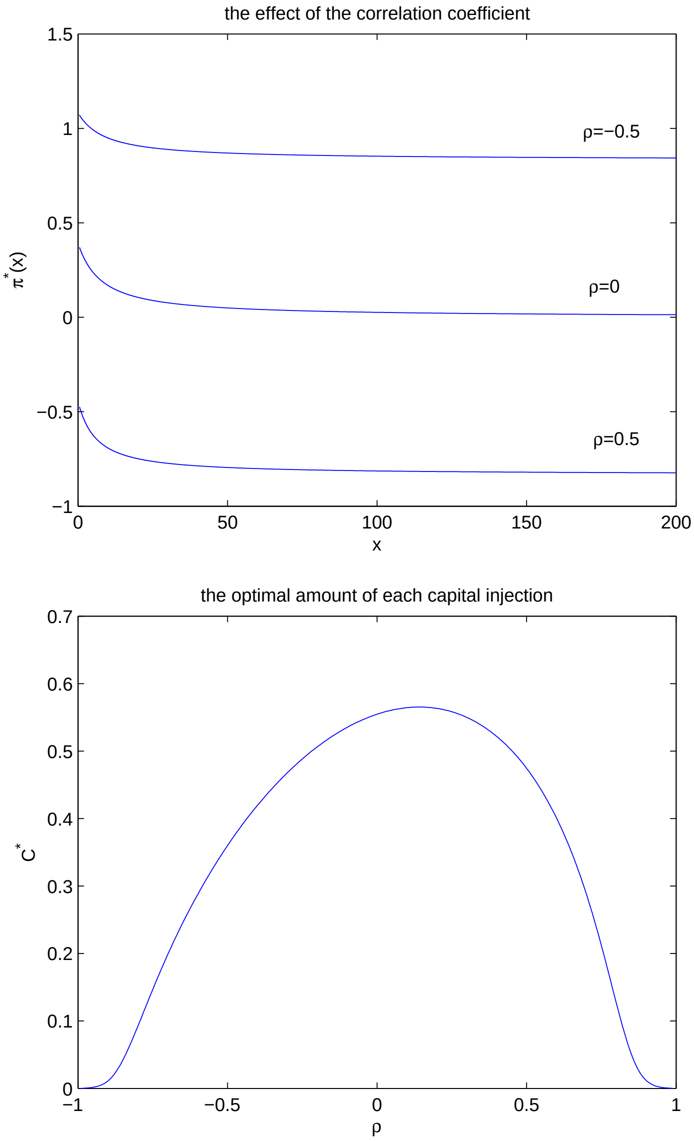


Figure 4: Effect of the risk-free interest rate r_0

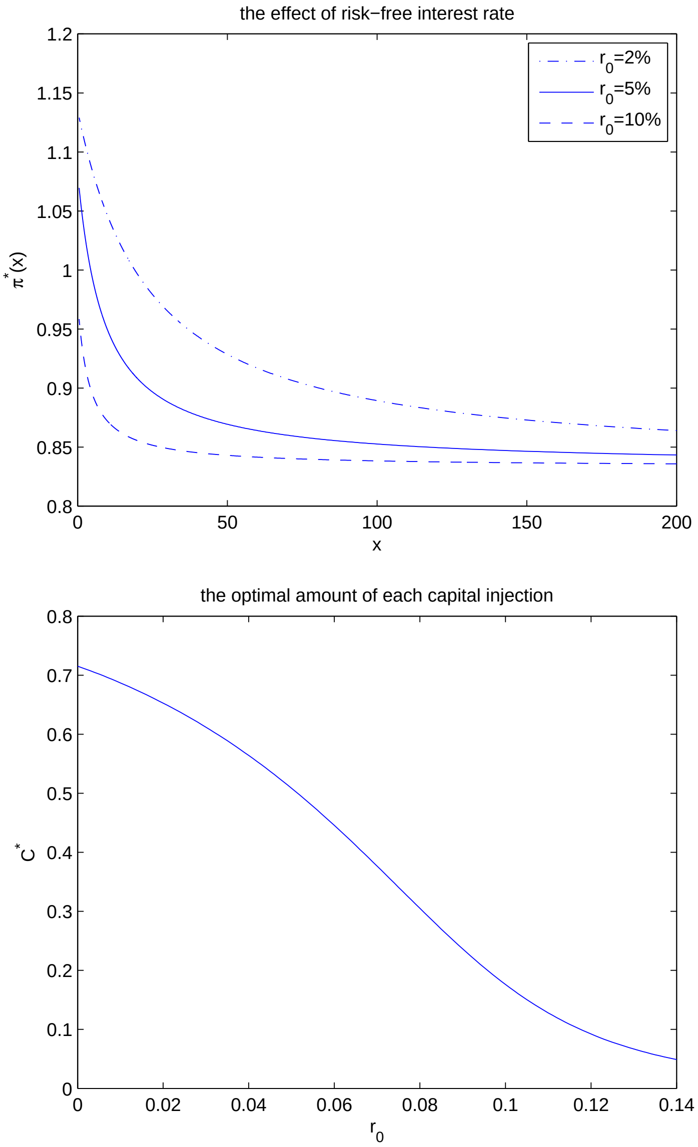
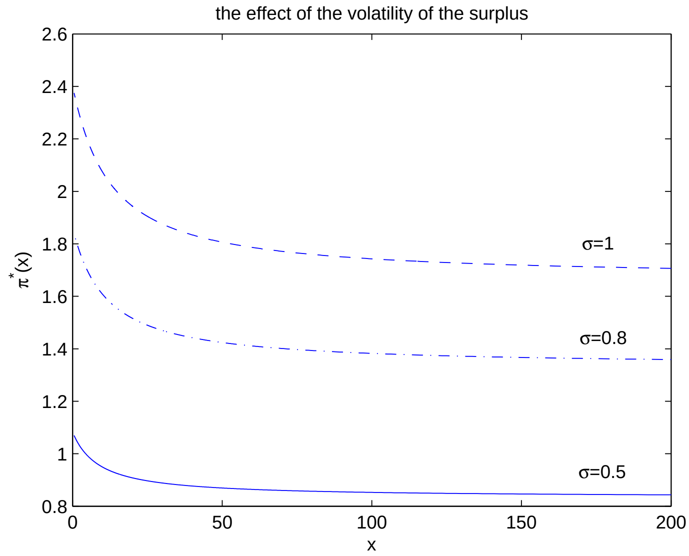


Figure 5: Effect of the surplus volatility σ



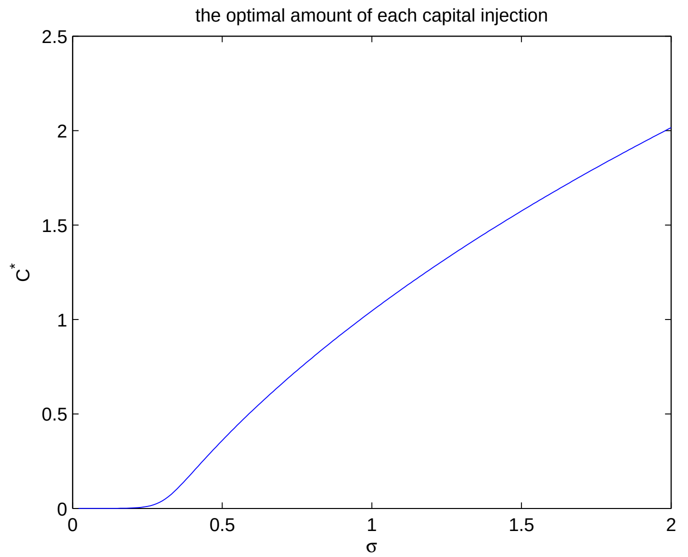
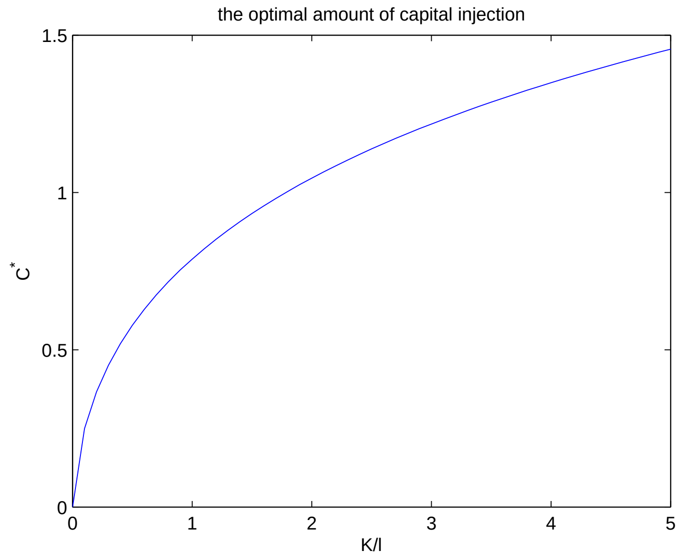


Figure 6: Effect of the transaction costs K and l



6 Concluding remarks

In this paper, we consider the optimal portfolio selection and capital injection policies which minimize the expected value of the total discounted capital injection costs in the particular case for which capital injections happen when the surplus is zero. The portfolio's surplus is allowed to be invested in the classical Black-Scholes financial market, and capital injection is assumed to be made when the surplus hits zero. This leads to a regular-impulse mixed stochastic control problem in a diffusion model. The main contribution of this paper is that we obtain explicit solutions for the optimal investment policy, the optimal capital injection policy, and the value function in three cases, namely, a perfect correlation between the risky asset and the surplus process, a zero risk-free interest rate, or a zero discount rate in the definition of the company's optimization problem. In other cases, an explicit expression for the optimal investment policy can only be found for values of the surplus account exceeding a certain threshold. In addition, we propose a method to solve the optimal investment policy numerically. By a change of variable, the optimal investment policy can be translated to the solution to an IVP of a singular ODE. Then, the backward Euler method is applied to solve the IVP numerically.

In the model set-up of this paper, there are no short shelling and borrowing restrictions for the investment and the surplus level for capital injection is fixed. The same optimization problem with investment restrictions but no capital injection restrictions is another interesting topic.

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